

Maximum Throughput in Multiple-Antenna Systems

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Abstract

The point-to-point multiple-antenna channel is investigated in uncorrelated block fading environment with Rayleigh distribution. The maximum throughput and maximum expected-rate of this channel are derived under the assumption that the transmitter is oblivious to the channel state information (CSI), however, the receiver has perfect CSI. First, we prove that in multiple-input single-output (MISO) channels, the optimum transmission strategy maximizing the throughput is to use all available antennas and perform equal power allocation with uncorrelated signals. Furthermore, to increase the expected-rate, multi-layer coding is applied. Analogously, we establish that sending uncorrelated signals and performing equal power allocation across all available antennas at each layer is optimum. A closed form expression for the maximum continuous-layer expected-rate of MISO channels is also obtained. Moreover, we investigate multiple-input multiple-output (MIMO) channels, and formulate the maximum throughput in the asymptotically low and high SNR regimes and also asymptotically large number of transmit or receive antennas by obtaining the optimum transmit covariance matrix. Finally, a distributed antenna system, wherein two single-antenna transmitters want to transmit a common message to a single-antenna receiver, is considered. It is shown that this system has the same outage probability and hence, throughput and expected-rate, as a point-to-point 2×1 MISO channel.

I. INTRODUCTION

The information theoretic aspects of wireless fading channels have received wide attention [1]. The growing demand for QoS and network coverage inspires the use of multiple-antenna arrays at the transmitter and/or receiver [2]–[5]. It has been shown that multiple-antenna arrays have the ability to reach higher transmission rates [6]–[8]. With no delay constraint, the ergodic nature of the fading channel can be experienced by sending very large transmission blocks, and the ergodic capacity is well studied [1]. When the channel variation is slow, the channel can be estimated relatively accurately at the receiver. By assuming perfect CSI at the receiver but no CSI at the transmitter, Telatar [6] showed that the ergodic

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capacity of general MIMO channels is achieved by sending an uncorrelated circularly symmetric zero mean equal power complex Gaussian codebook on all transmit antennas.

Due to the stringent delay constraint for the problem in consideration, the transmission block length is forced to be shorter than the dynamics of the slow fading process, though still large enough to yield a reliable communication. The performance of such channels are usually evaluated by outage capacity. The notion of capacity versus outage was introduced in [1], [9]. Jorswieck and Boch [10] proved that in uncorrelated MISO channels, the optimum transmit strategy minimizing the outage probability is to use a fraction of all available transmit antennas and perform equal power allocation with uncorrelated signals.

The maximum throughput is an important performance measure in block fading channels [11], which is defined as the maximum of the product of the transmission rate and the probability of successful transmission using a single-layer code (see Definition 1). As mentioned in [10], their results on the outage probability cannot be directly applied to this metric due to the maximization. In this paper, we prove that to achieve the maximum throughput in an uncorrelated MISO channel, the optimum transmit strategy is to send equal power uncorrelated signals from all available antennas (see Theorem 1).

The maximum average achievable rate is another performance measure which is important in some applications. A good example for such applications is a TV broadcasting system where users with better channels can receive additional services such as high definition TV signals [12]. Due to the large number of users, the transmitter cannot access the CSI. In order to increase the average achievable rate, Shamaï and Steiner [13] proposed a broadcast approach (multi-layer coding) for a point-to-point block fading channel with no CSI at the transmitter. Since the average achievable rate increases with the number of code layers, they reached the highest average achievable rate using a continuous-layer (infinite-layer) code. This idea was applied to a two-hop single-relay channel in [14], [15], a channel with two collocated cooperative users in [16], and a two-hop parallel-relay network (the diamond channel) in [17]. Multi-layer coding can also achieve the maximum average achievable rate in a block fading multiple-access channel with no CSI at the transmitters [18]. The optimized trade-off between the QoS and network coverage in a multicast network was derived in [12] using the broadcast approach. Here, we derive the maximum expected-rate of MISO channels, which is defined as the maximum average decodable rate when a multi-layer code is transmitted (see Definition 2). Theorem 2 proves that to maximize the expected-rate in MISO channels, it is optimum to transmit equal power independent signals on all available antennas in each layer. Using the continuous-layer coding approach, the maximum expected-rate of MISO channels is then obtained and formulated in closed form in Proposition 4.

To evaluate the maximum throughput in uncorrelated MIMO channels, the distribution of the instantaneous mutual information is crucial. In [19], [20], it is shown that the distribution of the instantaneous

mutual information in MIMO channels is always very close to the Gaussian distribution. The mean and variance of this equivalent Gaussian distribution were derived in [20] for asymptotic ranges of the number of antennas. As this distribution is not tractable in general MIMO channels, here we consider four asymptotic cases: asymptotically low SNR regime, asymptotically high SNR regime, asymptotically large number of transmit antennas, and asymptotically large number of receive antennas. In all four cases, the optimum covariance matrix is obtained and the maximum throughput expression is derived.

Finally, the maximum throughput and maximum expected-rate of a distributed antenna system with two single-antenna transmitters and one single-antenna receiver is obtained. It is also proved that any achievable throughput, expected-rate, ergodic capacity, and outage capacity in a MISO channel with two transmit antennas are also achievable in this channel.

The rest of this paper is organized as follows. In Section II, the preliminaries are presented. The maximum throughput and the maximum expected-rate of MISO channels are derived in Sections III and IV, respectively. The maximum throughputs in four asymptotic cases of MIMO channels are obtained in Section V. In Section VI, a distributed antenna system with two transmitters is analyzed. Finally, Section VII concludes the paper.

II. PRELIMINARIES

A. Notation

Throughout the paper, we represent the probability of event A by $\Pr\{A\}$, and the expected and variance operations by $\mathbb{E}(\cdot)$ and $\text{Var}(\cdot)$, respectively. The notation “ \ln ” is used for natural logarithm, and rates are expressed in *nats*. We denote $f_x(\cdot)$ and $F_x(\cdot)$ as the probability density function (PDF) and the cumulative density function (CDF) of random variable x , respectively. For any function $F(x)$, let us define $\overline{F}(x) \triangleq 1 - F(x)$ and $F'(x) \triangleq \frac{dF(x)}{dx}$. \vec{X} is a vector, \mathbf{Q} is a matrix, and $\text{tr}(\mathbf{Q})$ denotes the trace of \mathbf{Q} . \mathbf{I}_{n_t} denotes the $n_t \times n_t$ identity matrix. s^o is the optimum solution with respect to the variable s . We denote the conjugation, matrix transpose, and matrix conjugate transpose operators by $*$, T , and † , respectively. $\Re(\cdot)$ and $\Im(\cdot)$ represent the real and imaginary parts of complex variables and $|\cdot|$ represents the absolute value or modulus operator. “ \det ” is used for the determinant operator and $\text{eig}_\ell(\mathbf{Q})$ is the ℓ 'th ordered eigenvalue of matrix \mathbf{Q} . Let h_ℓ denote the ℓ 'th component of vector \vec{h} , and $h_{\ell,k}$ denote the (ℓ, k) 'th entry of matrix \mathbf{H} . $\mathcal{CN}(0, 1)$ denotes the complex circularly symmetric Gaussian distribution with zero mean and unit variance and $\mathcal{N}(\mu, \sigma^2)$ denotes the Gaussian distribution with mean μ and variance σ^2 . $\mathcal{W}_0(\cdot)$ is the zero branch of the Lambert W -function, also called the omega function, which is the inverse function of $f(W) = We^W$ [21]. $E_1(x)$ is the exponential integral function, which is $\int_x^\infty \frac{e^{-t}}{t} dt$, $x \geq 0$. $\Gamma(n, x) \triangleq \int_x^\infty t^{n-1} e^{-t} dt$ is the upper incomplete gamma function, and $\Gamma(n) \triangleq \Gamma(n, 0)$. $F(n) \triangleq \frac{\Gamma'(n)}{\Gamma(n)}$ and

$\mathcal{Q}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$ represent the Euler's digamma function [22] and \mathcal{Q} -function, respectively.

B. Problem Setup

A MIMO channel with n_t transmit antennas and n_r receive antennas is defined as a channel with the following input-output relationship:

$$\vec{Y} = \mathbf{H}\vec{X} + \vec{Z}, \quad (1)$$

where \vec{Y} is the received signal, $\mathbf{H} \sim [\mathcal{CN}(0, 1)]_{n_r \times n_t}$ is the channel matrix, $\vec{Z} \sim [\mathcal{CN}(0, 1)]_{n_r \times 1}$ is the independent and identically distributed (i.i.d.) additive white Gaussian noise (AWGN), and \vec{X} is the transmitted signal under the following total power constraint:

$$\mathbb{E}(\vec{X}^\dagger \vec{X}) = \mathbb{E}(\text{tr}(\vec{X} \vec{X}^\dagger)) = \text{tr}(\mathbb{E}(\vec{X} \vec{X}^\dagger)) \leq P. \quad (2)$$

Defining \mathbf{Q} as the transmit covariance matrix, i.e., $\mathbf{Q} = \mathbb{E}(\vec{X} \vec{X}^\dagger)$, the instantaneous mutual information is

$$\mathcal{I} = \ln \det(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger) = \ln \det(\mathbf{I}_{n_t} + \mathbf{Q}\mathbf{H}^\dagger\mathbf{H}). \quad (3)$$

In a MISO channel, the channel coefficients are represented by a vector $\vec{h}^T \sim [\mathcal{CN}(0, 1)]_{n_t \times 1}$, and

$$Y = \vec{h}\vec{X} + Z. \quad (4)$$

In the following, the performance metrics which are widely used throughout the paper are defined.

Definition 1 The throughput \mathcal{R}_s is the average achievable rate when a single-layer code with a fixed rate R is transmitted, i.e., the transmission rate times the probability of successful transmission. The maximum throughput, namely \mathcal{R}_s^m , is the maximum of the throughput over all transmit covariance matrices \mathbf{Q} , and transmission rates R . Mathematically,

$$\mathcal{R}_s^m \triangleq \max_{\substack{R, \mathbf{Q} \\ \text{tr}(\mathbf{Q}) \leq P}} \Pr\{\mathcal{I} \geq R\} R. \quad (5)$$

Definition 2 The expected-rate \mathcal{R}_f is the average achievable rate when a multi-layer code is transmitted, i.e., the statistical expectation of the achievable rate. The maximum expected-rate, namely \mathcal{R}_f^m , is the maximum of the expected-rate over all transmit covariance matrices and transmission rates in each layer, and all power distributions of the layers. Mathematically,

$$\mathcal{R}_f^m \triangleq \max_{\substack{R_i, P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^K \Pr\{\mathcal{I}_i \geq R_i\} R_i, \quad (6)$$

where R_i , \mathbf{Q}_i , and \mathcal{I}_i are the transmission rate, transmit covariance matrix, and instantaneous mutual information in the i 'th layer, respectively.

If a continuum of code layers are transmitted, the maximum continuous-layer (infinite-layer) expected-rate, namely \mathcal{R}_c^m , is given by maximizing the continuous-layer expected-rate over the layers' power distribution.

Definition 3 The ergodic capacity C_{erg} is the maximum expected value of the instantaneous mutual information \mathcal{I} over all transmit covariance matrices \mathbf{Q} . Mathematically,

$$C_{\text{erg}} \triangleq \max_{\substack{\mathbf{Q} \\ \text{tr}(\mathbf{Q}) \leq P}} \mathbb{E}(\mathcal{I}). \quad (7)$$

The main focus of this paper is to solve the following problems.

Problem 1 To obtain the optimum transmit covariance matrix, denoted by \mathbf{Q}^o , which maximizes the throughput \mathcal{R}_s in the MISO channel.

Theorem 1 proves that the optimum transmit strategy is to transmit uncorrelated signals on all antennas with equal powers, i.e., $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$, and provides the maximum throughput expression.

Problem 2 To derive the optimum transmit covariance matrix in each layer, i.e., \mathbf{Q}_i^o , for finite-layer coding in the MISO channel, which maximizes the expected-rate \mathcal{R}_f .

As we shall see in Theorem 2, the optimum transmit covariance matrix in each layer is in the form of $\mathbf{Q}_i^o = \frac{P_i}{n_t} \mathbf{I}_{n_t}$, and the maximum expected-rate is given by Eq. (32).

Problem 3 To derive the maximum continuous-layer expected-rate \mathcal{R}_c^m in the MISO channel.

The closed form expression of the maximum continuous-layer expected-rate is derived in Proposition 4.

In the MIMO channel, the PDF of the instantaneous mutual information \mathcal{I} is not known even for the simplest case of $\mathbf{Q} = \frac{P}{n_t} \mathbf{I}_{n_t}$, although there are some approximations in literature for asymptotic cases. In the next step, the maximum throughputs in four asymptotic cases of the MIMO channel are addressed.

Problem 4 To derive the maximum throughput of the MIMO channel in asymptotically

- low SNR regime
- high SNR regime
- large number of transmit antennas
- large number of receive antennas

Different MIMO approximations are exploited to solve Problem 4. For asymptotically low SNR regime, the MISO results are carried over and the maximum throughput and maximum expected-rate are formulated. For asymptotically high SNR regime, Wishart distribution properties [23] are used to obtain the maximum throughput. For asymptotically large number of transmit or receive antennas, Gaussian approximations for the instantaneous mutual information presented in [20] are utilized. As we shall see in Section V, in all aforementioned asymptotic regimes, the optimum transmit covariance matrix which maximizes the throughput is $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$.

In the last problem, a distributed antenna system consisting of two single-antenna transmitters with common messages and a single-antenna receiver is considered.

Problem 5 *To find the minimum outage probability, the maximum throughput, and the maximum expected-rate in a two-transmitter distributed antenna system.*

Theorem 6 establishes that any achievable outage probability in the 2×1 MISO channel is also achievable in the two-transmitter distributed antenna system in Problem 5. Hence, both channels experience the same instantaneous mutual information distribution and thereby, all MISO channel results are applied here with $n_t = 2$.

C. A Few Useful Propositions

In the following, we present three propositions which are used throughout the paper and they are also of independent interest.

Proposition 1 *In fading channels, the maximum throughput is less than or equal to the ergodic capacity.*

Proof: The proof is based on the Markov inequality [24], that is if $f(x) = 0$ for $x < 0$, then, for $\alpha > 0$, $\Pr \{x \geq \alpha\} \leq \frac{\mathbb{E}(x)}{\alpha}$. Therefore, $\forall R > 0$,

$$\Pr \{\mathcal{I} \geq R\} \leq \frac{\mathbb{E}(\mathcal{I})}{R}, \quad (8)$$

so that

$$\mathcal{R}_s^m = \max_{R, \mathbf{Q}} \Pr \{\mathcal{I} \geq R\} R \leq \max_{\substack{\mathbf{Q} \\ \text{tr}(\mathbf{Q}) \leq P}} \mathbb{E}(\mathcal{I}), \quad (9)$$

and Eq. (9) results because $\max_{\mathbf{Q}, \text{tr}(\mathbf{Q}) \leq P} \mathbb{E}(\mathcal{I})$ equals the ergodic capacity. ■

Proposition 2 *In fading channels, the maximum expected-rate is less than or equal to the ergodic capacity.*

Proof: From Eq. (6) it follows that

$$\begin{aligned}
\mathcal{R}_f^m &= \max_{\substack{R_i, P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^K \Pr \{ \mathcal{I}_i \geq R_i \} R_i \\
&\stackrel{(a)}{\leq} \max_{\substack{P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^K \mathbb{E} (\mathcal{I}_i) \\
&\stackrel{(b)}{=} \max_{\substack{P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E} \left(\sum_{i=1}^K \mathcal{I}_i \right) \\
&= \max_{\substack{P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E} \left(\sum_{i=1}^K \ln \frac{\det \left(\mathbf{I}_{n_t} + \sum_{j=i}^K \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H} \right)}{\det \left(\mathbf{I}_{n_t} + \sum_{j=i+1}^K \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H} \right)} \right) \\
&= \max_{\substack{P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E} \left(\ln \prod_{i=1}^K \frac{\det \left(\mathbf{I}_{n_t} + \sum_{j=i}^K \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H} \right)}{\det \left(\mathbf{I}_{n_t} + \sum_{j=i+1}^K \mathbf{Q}_j \mathbf{H}^\dagger \mathbf{H} \right)} \right) \\
&= \max_{\substack{P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \mathbb{E} \left(\ln \det \left(\mathbf{I}_{n_t} + \sum_{i=1}^K \mathbf{Q}_i \mathbf{H}^\dagger \mathbf{H} \right) \right), \tag{10}
\end{aligned}$$

where (a) follows from Proposition 1, and (b) follows from the fact that expectation and summation commute. Defining $\mathbf{Q} \triangleq \sum_{i=1}^K \mathbf{Q}_i$, we get

$$\text{tr}(\mathbf{Q}) = \text{tr} \left(\sum_{i=1}^K \mathbf{Q}_i \right) = \sum_{i=1}^K \text{tr}(\mathbf{Q}_i) \leq \sum_{i=1}^K P_i = P. \tag{11}$$

Inserting Eq. (11) into Eq. (10), we obtain

$$\mathcal{R}_f^m \leq \max_{\substack{\mathbf{Q} \\ \text{tr}(\mathbf{Q}) \leq P}} \mathbb{E} \left(\ln \det \left(\mathbf{I}_{n_t} + \mathbf{Q} \mathbf{H}^\dagger \mathbf{H} \right) \right) = \max_{\substack{\mathbf{Q} \\ \text{tr}(\mathbf{Q}) \leq P}} \mathbb{E} (\mathcal{I}). \tag{12}$$

and Eq. (12) results because $\max_{\mathbf{Q}, \text{tr}(\mathbf{Q}) \leq P} \mathbb{E} (\mathcal{I})$ equals the ergodic capacity. ■

Propositions 1 and 2 lead to the fact that the maximum throughput and maximum expected-rate are upper-bounded by the ergodic capacity. Proposition 3 presents the ergodic capacity of the MISO channel in closed form.

Proposition 3 *The ergodic capacity in an $n_t \times 1$ MISO Rayleigh fading channel with total power constraint*

P is given by

$$C_{\text{erg}} = e^{\frac{n_t}{P}} \mathbf{E}_1 \left(\frac{n_t}{P} \right) \sum_{\ell=0}^{n_t-1} \frac{(-n_t)^\ell}{\ell! P^\ell} + \sum_{\ell=1}^{n_t-1} \sum_{k=0}^{\ell-1} \frac{(-1)^k}{(\ell-k) k!} \sum_{m=0}^{\ell-k-1} \frac{(n_t)^{k+m}}{m! P^{k+m}}, \quad (13)$$

where $\mathbf{E}_1(\cdot)$ is the exponential integral function. The ergodic capacity in a $1 \times n_r$ single-input multiple-output (SIMO) channel with total power constraint P equals the ergodic capacity of an $n_r \times 1$ MISO channel with total power constraint $n_r P$.

Proof: We offer the proof in appendix A. ■

III. MAXIMUM THROUGHPUT IN MISO CHANNELS

Let the transmitted signal \vec{X} be a single-layer code with rate $R = \ln(1 + Ps)$. In the MISO channel, the maximum throughput in Eq. (5) can be rewritten as

$$\mathcal{R}_s^m = \max_{\substack{R, \mathbf{Q} \\ \text{tr}(\mathbf{Q}) \leq P}} \Pr \left\{ \ln \left(1 + \vec{h} \mathbf{Q} \vec{h}^\dagger \right) \geq R \right\} R, \quad (14)$$

where \mathbf{Q} is the covariance matrix of \vec{X} , i.e., $\mathbf{Q} = \mathbb{E} \left(\vec{X} \vec{X}^\dagger \right)$.

For transmission rate R , the throughput is $\mathcal{R}_s = \overline{\mathcal{P}}_{\text{out}}(R) R$, where $\mathcal{P}_{\text{out}}(R)$ is the outage probability of a fixed transmission rate R . It is proved in [10] that the optimum transmit strategy minimizing the outage probability is to send uncorrelated circularly symmetric zero mean equal power complex Gaussian signals from a fraction of antennas. Thus, here, one can restrict the transmit covariance matrix \mathbf{Q} to diagonal matrices whose diagonal entries are either zero or a constant subject to the total power constraint P .

In following, Theorem 1 proves that the optimum solution with respect to R , denoted by R^o , maximizing $\overline{\mathcal{P}}_{\text{out}}(R) R$ is less than $\ln(1 + P)$. In this range of the transmission rate, the optimum transmit strategy which minimizes the outage probability and consequently, maximizes the throughput is to use all available antennas. Equation (15) yields the maximum throughput of an $n_t \times 1$ MISO block Rayleigh fading channel.

Theorem 1 *In a single-layer $n_t \times 1$ MISO block Rayleigh fading channel, the optimum transmit covariance matrix which maximizes the throughput is $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$. The maximum throughput is given by*

$$\mathcal{R}_s^m = \max_{0 < s < 1} \frac{\Gamma(n_t, n_t s)}{(n_t - 1)!} \ln(1 + Ps). \quad (15)$$

Proof:

As pointed out above, we can restrict our attention to assume that l_t out of n_t transmit antennas are active and perform equal power allocation. Equation (14) is simplified to

$$\begin{aligned}
\mathcal{R}_s^m &= \max_{R, l_t} \Pr \left\{ \ln \left(1 + \frac{P}{l_t} \sum_{\ell=1}^{l_t} |h_\ell|^2 \right) \geq R \right\} R \\
&= \max_{s, l_t} \Pr \left\{ \sum_{\ell=1}^{l_t} |h_\ell|^2 \geq l_t s \right\} R \\
&= \max_{s, l_t} \overline{F}_a(l_t s) \ln(1 + Ps),
\end{aligned} \tag{16}$$

where $a \triangleq \sum_{\ell=1}^{l_t} |h_\ell|^2$ is gamma-distributed and thereby, $\overline{F}_a(x) = \frac{\Gamma(l_t, x)}{\Gamma(l_t)}$. The first derivative of $\mathcal{R}_s(s) = \overline{F}_a(l_t s) \ln(1 + Ps)$ with respect to s is

$$\mathcal{R}'_s(s) = \overline{F}_a(l_t s) \frac{P}{1 + Ps} - l_t f_a(l_t s) \ln(1 + Ps). \tag{17}$$

Let us define the following functions,

$$r(s) \triangleq \frac{\overline{F}_a(l_t s)}{l_t f_a(l_t s)}, \tag{18}$$

$$g(s, P) \triangleq \ln(1 + Ps)^{\frac{1+Ps}{P}}. \tag{19}$$

As such, we get

$$\begin{cases} \mathcal{R}'_s(s) > 0 & \text{iff } r(s) > g(s, P), \\ \mathcal{R}'_s(s) = 0 & \text{iff } r(s) = g(s, P), \\ \mathcal{R}'_s(s) < 0 & \text{iff } r(s) < g(s, P). \end{cases} \tag{20}$$

Noting $\overline{F}_a(x) = \frac{\Gamma(l_t, x)}{\Gamma(l_t)}$ and $f_a(x) = \frac{x^{l_t-1} e^{-x}}{\Gamma(l_t)}$, we have

$$r(s) = \frac{\Gamma(l_t, l_t s)}{l_t (l_t s)^{l_t-1} e^{-l_t s}} = \frac{\Gamma(l_t, l_t s)}{l_t^l s^{l_t-1} e^{-l_t s}}. \tag{21}$$

For positive integer arguments of m , $\Gamma(m, x) = (m-1)! e^{-x} \sum_{\ell=0}^{m-1} \frac{x^\ell}{\ell!}$. Inserting the above equation into Eq. (21) yields

$$\begin{aligned}
r(s) &= \frac{(l_t - 1)! e^{-l_t s} \sum_{\ell=0}^{l_t-1} \frac{(l_t s)^\ell}{\ell!}}{l_t (l_t s)^{l_t-1} e^{-l_t s}} \\
&= \frac{1}{l_t} + \frac{1}{l_t} \sum_{\ell=0}^{l_t-2} \frac{(l_t - 1) \dots (\ell + 1)}{(l_t s)^{l_t-\ell-1}} \\
&= \frac{1}{l_t} + \frac{1}{l_t} \sum_{\ell=0}^{l_t-2} \prod_{k=0}^{l_t-\ell-2} \frac{l_t - k - 1}{l_t s}.
\end{aligned} \tag{22}$$

As $\frac{l_t - k - 1}{l_t s} < 1$ for $s \geq 1$, replacing in Eq. (22) gives

$$r(s) \leq \frac{1}{l_t} + \frac{1}{l_t} \sum_{\ell=0}^{l_t-2} \prod_{k=0}^{l_t-\ell-2} 1 = \frac{1}{l_t} + \frac{l_t - 1}{l_t} = 1, \quad \forall s \geq 1. \tag{23}$$

From Eq. (22), $\lim_{s \rightarrow 0} r(s) = +\infty$.

On the other hand, the first derivative of $g(s)$ with respect to P is

$$\begin{aligned} \frac{\partial g(s, P)}{\partial P} &= \frac{sP - \ln(1 + sP)}{P^2} \\ &= \frac{1}{P^2} \ln \frac{e^{sP}}{1 + sP} \\ &= \frac{1}{P^2} \ln \left(1 + \frac{1}{1 + sP} \sum_{k=2}^{\infty} \frac{(sP)^k}{k!} \right) > 0. \end{aligned} \quad (24)$$

Therefore, $g(s, P)$ is a strictly increasing function with respect to P . As a result,

$$g(s, P) > \lim_{P \rightarrow 0} \ln(1 + Ps)^{\frac{1+Ps}{P}} = s. \quad (25)$$

Comparing Eq. (23), Eq. (25), $\lim_{s \rightarrow 0} r(s) = +\infty$, and $g(0, P) = 0$, we get

$$\begin{cases} r(s) > g(s, P) & s = 0, \\ r(s) < g(s, P) & s \geq 1. \end{cases} \quad (26)$$

Inserting Eq. (26) into Eq. (20) yields

$$\begin{cases} \mathcal{R}'_s(s) > 0 & s = 0, \\ \mathcal{R}'_s(s) < 0 & s \geq 1. \end{cases} \quad (27)$$

Since $\mathcal{R}_s(s)$ is a continuous function, according to Eq. (27), for all positive integer values of l_t and positive values of P , one can conclude that $\mathcal{R}_s(s)$ takes its maximum at $0 < s^o < 1$.

Jorswieck and Boche [10] proved that when $P > e^R - 1$, or equivalently $s < 1$, the optimum transmission strategy to minimize the outage probability is to use all available antennas with equal power allocation. Since $\forall l_t, 0 < s^o < 1$, the optimum strategy maximizing the throughput is to use all available antennas and perform equal power allocation. The maximum throughput is given by Eq. (15). ■

Remark 1 In point-to-point single-input single-output (SISO) channels, by substituting $n_t = 1$ in Eq. (15), the optimum solution with respect to s is $s^o = \frac{1}{\mathcal{W}_0(P)} - \frac{1}{P}$, where $\mathcal{W}_0(\cdot)$ is the zero branch of the Lambert W-function. Therefore,

$$\mathcal{R}_s^m = e^{\frac{1}{P} - \frac{1}{\mathcal{W}_0(P)}} \ln \left(\frac{P}{\mathcal{W}_0(P)} \right). \quad (28)$$

From Proposition 3, the ergodic capacity in this channel is

$$C_{\text{erg}} = e^{\frac{1}{P}} \mathbf{E}_1 \left(\frac{1}{P} \right). \quad (29)$$

Remark 2 Note that $g(s, P)$ is a strictly increasing function with respect to s and P , and $r(s)$ is a strictly decreasing function with respect to s and increases with the number of transmit antennas. Therefore, the solution to $r(s) = g(s, P)$, i.e., s^o ,

- decreases with P . In asymptotically high SNR regime, $s^o \rightarrow 0$.
- increases with n_t . In asymptotically large number of transmit antennas, $s^o \rightarrow 1$.

As a byproduct result of Theorem 1 and remark 2, we have the following.

Corollary 1 In the asymptotically large number of transmit antennas MISO channel, the maximum throughput is given by

$$\mathcal{R}_s^m = \lim_{s \rightarrow 1} \frac{\Gamma(n_t, n_t s)}{(n_t - 1)!} \ln(1 + Ps) \xrightarrow{n_t \rightarrow \infty} \ln(1 + P). \quad (30)$$

Remark 3 In a correlated MISO channel wherein the transmitter does neither know the CSI nor the channel correlation, the outage probability is a Schur-convex (resp. Schur-concave) function of the channel covariance matrix for $P > e^R - 1$ (resp. $P < \frac{e^R - 1}{2}$) [10]. According to Theorem 1, in the maximum throughput of the MISO channel, i.e., $\bar{\mathcal{P}}_{out}(R^o)R^o$, we have $e^{R^o} - 1 < P$. Hence, in this range of the transmission rate, \mathcal{R}_s is a Schur-concave function of the channel covariance matrix, i.e., channel correlation decreases the throughput. In terms of the impact of correlation in the MISO channel with no CSI at the transmitter, the behavior of the maximum throughput is similar to the behavior of the ergodic capacity which is also a Schur-concave function of the channel covariance matrix [25].

IV. MAXIMUM EXPECTED-RATE IN MISO CHANNELS

A block fading channel can be modeled by an equivalent broadcast channel whose receiver channels represent any fading coefficient realization. The expected-rate of a fading channel is equal to a weighted sum-rate of its equivalent broadcast channel in which the weights distribution is the complementary CDF (tail distribution) of the channel gain [26]. In broadcast channels, any maximum weighted sum-rate with positive value weights is on the capacity region [12]. Since superposition (multi-layer) coding achieves the capacity region of degraded broadcast channels [27], it is the optimum coding strategy to maximize the average achievable rate in any block fading channel whose equivalent broadcast channel is degraded [13]. An example for such channels is the SISO channel. Although multi-layer coding is not the optimum coding strategy in MISO channels, it increases the average achievable rate of the channel. Numerical results for the continuous-layer expected-rate of MISO and SIMO block Rayleigh fading channels were presented in [28]. Here, the optimum transmit covariance matrix at each code layer is obtained, and consequently, the maximum expected-rate of the MISO channel is analytically formulated. Note that the

maximum expected-rate of the SIMO channel can be calculated using the same formula by replacing P with $n_t P$ in Eq. (42).

In order to enhance the lucidity of this section, we divide it into two subsections. Section IV-A presents the maximum expected-rate of the MISO channel when a finite-layer code is transmitted. The more code layers, the higher expected-rate. Hence, a continuous-layer (infinite-layer) code yields the highest expected-rate of the channel. The maximum continuous-layer expected-rate of the MISO channel is derived in Section IV-B in closed form.

A. Finite-Layer Code

In finite-layer coding approach, the transmitter sends a K -layer code $\vec{X} = \sum_{i=1}^K \vec{X}_i$. Let P_i be the signal power in the i 'th layer with rate $R_i = \ln \left(1 + \frac{P_i s_i}{1 + I_i s_i} \right)$, where $I_i = \sum_{j=i+1}^K P_j$ is the power of the upper layers while decoding the i 'th layer. The maximum expected-rate in Eq. (6) is simplified to

$$\mathcal{R}_f^m = \max_{\substack{R_i, P_i, \mathbf{Q}_i \\ \text{tr}(\mathbf{Q}_i) \leq P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^K \Pr \left\{ \ln \left(1 + \frac{\vec{h} \mathbf{Q}_i \vec{h}^\dagger}{\vec{h} \sum_{j=i+1}^K \mathbf{Q}_j \vec{h}^\dagger} \right) \geq R_i \right\} R_i. \quad (31)$$

Theorem 2 presents the optimum covariance matrix in each layer which maximizes the expected-rate in the MISO channel.

Theorem 2 *In a finite-layer $n_t \times 1$ MISO block Rayleigh fading channel, the optimum transmit covariance matrix in each layer which maximizes the expected-rate is $\mathbf{Q}_i^o = \frac{P_i}{n_t} \mathbf{I}_{n_t}$, where P_i is the power allocated to the i 'th layer. The maximum K -layer expected-rate is given by*

$$\mathcal{R}_f^m = \max_{\substack{0 < s_i < 1, P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^K \frac{\Gamma(n_t, n_t s_i)}{(n_t - 1)!} \ln \left(1 + \frac{P_i s_i}{1 + \sum_{j=i+1}^K P_j s_j} \right). \quad (32)$$

Proof: Since the outage probability does not depend on the directions of the transmit covariance matrix \mathbf{Q} [29], the problem is diagonalized. Therefore, the expected-rate received at the destination is simplified to

$$\mathcal{R}_f = \sum_{i=1}^K \Pr \left\{ \ln \left(1 + \frac{P_i \sum_{\ell=1}^{n_t} \delta_\ell |h_\ell|^2}{1 + I_i \sum_{\ell=1}^{n_t} \eta_\ell |h_\ell|^2} \right) \geq R_i \right\} R_i, \quad (33)$$

where δ_ℓ and η_ℓ are the power fraction and upper-layer interference portion at the ℓ 'th antenna, respectively, subject to $\sum_{\ell=1}^{n_t} \delta_\ell = \sum_{\ell=1}^{n_t} \eta_\ell = 1$. Equation (33) can be rewritten as

$$\mathcal{R}_f = \sum_{i=1}^K \Pr \left\{ \sum_{\ell=1}^{n_t} (\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell) |h_\ell|^2 \geq s_i \right\} R_i. \quad (34)$$

As $\sum_{\ell=1}^{n_t} (\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell) = 1$, to minimize $\Pr \{ \sum_{\ell=1}^{n_t} (\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell) |h_\ell|^2 < s_i \}$, $\forall i$, the optimum value of $\delta_\ell + s_i I_i \delta_\ell - s_i I_i \eta_\ell$ must be either zero or a constant independent of ℓ for any positive value

of s_i . Hence, up to now, the optimum solution to Eq. (34) is to choose either $\delta_\ell = \eta_\ell = \frac{1}{l_{t_i}}$ or $\delta_\ell = \eta_\ell = 0$, that is to use l_{t_i} out of n_t antennas with power $\frac{P_i}{l_{t_i}}$ in each layer. Therefore, Eq. (34) is simplified to

$$\mathcal{R}_f = \sum_{i=1}^K \Pr \left\{ \sum_{\ell=1}^{l_{t_i}} |h_\ell|^2 \geq l_{t_i} s_i \right\} R_i = \sum_{i=1}^K \bar{F}_{a_i}(l_{t_i} s_i) R_i, \quad (35)$$

where $a_i = \sum_{\ell=1}^{l_{t_i}} |h_\ell|^2$. In the remainder of the proof, we shall show that the optimum solution with respect to l_{t_i} is $l_{t_i}^o = n_t$, $\forall i$. Analogous to the throughput case in Theorem 1, let us define

$$\mathcal{R}_s(s_i) \triangleq \bar{F}_{a_i}(l_{t_i} s_i) \ln \left(1 + \frac{P_i s_i}{1 + I_i s_i} \right), \quad (36)$$

$$r(s_i) \triangleq \frac{\bar{F}_{a_i}(l_{t_i} s_i)}{l_{t_i} f_{a_i}(l_{t_i} s_i)}, \quad (37)$$

$$g(s_i, P_i, I_i) \triangleq \frac{(1 + I_i s_i) (1 + (I_i + P_i) s_i)}{P_i} \ln \left(1 + \frac{P_i s_i}{1 + I_i s_i} \right). \quad (38)$$

Note that $g(0, P_i, I_i) = 0$, $\lim_{s_i \rightarrow 0} r(s_i) = +\infty$, and Eqs. (20) and (23) still hold by redefining $\mathcal{R}_s(s_i)$, $r(s_i)$, and $g(s_i, P_i, I_i)$ as above, and with s replaced by s_i .

Defining $\hat{P}_i \triangleq \frac{P_i}{1 + I_i s_i}$, from Eq. (25) and noting $I_i s_i \geq 0$, we have

$$\begin{aligned} g(s_i, P_i, I_i) &= (1 + I_i s_i) \frac{\left(1 + \frac{P_i s_i}{1 + I_i s_i} \right)}{\frac{P_i}{1 + I_i s_i}} \ln \left(1 + \frac{P_i s_i}{1 + I_i s_i} \right) \\ &\geq \ln \left(1 + \hat{P}_i s_i \right)^{\frac{(1 + \hat{P}_i s_i)}{\hat{P}_i}} > s_i, \quad \forall s_i \geq 1. \end{aligned} \quad (39)$$

Therefore, Eqs. (26) and (27) still hold with the above functions, and lead to $0 < s_i^o < 1$. This directly corresponds to the proof of Theorem 1 and shows that the optimum power allocation strategy is to use all available antennas with equal power allocation in each layer, i.e., $\mathbf{Q}_i^o = \frac{P_i}{n_t} \mathbf{I}_{n_t}$, and the maximum expected-rate is given by Eq. (32). ■

B. Continuous-Layer Code

In the continuous-layer coding, a.k.a. broadcast approach, a continuum of code layers is transmitted. Similar to finite-layer coding in Section IV-A, the receiver decodes the signal from the lowest layer up to the layer that the channel condition allows.

Proposition 4 yields a closed form expression for the maximum continuous-layer expected-rate in the MISO channel by optimizing the power distribution over the layers.

Proposition 4 *In the MISO block Rayleigh fading channel, the maximum continuous-layer expected-rate obtained by optimizing the power distribution over the layers is given by*

$$\mathcal{R}_c^m = \mathcal{R}(s_1) - \mathcal{R}(s_0), \quad (40)$$

where,

$$\begin{aligned} \mathcal{R}(s) = e^{-s} \sum_{\ell=1}^{n_t-1} \frac{1}{\ell!} \left(s^\ell - (n_t + 1 - \ell)(\ell - 1)! \sum_{k=0}^{\ell-1} \frac{s^k}{k!} \right) \\ + e^{-s} - (n_t + 1)E_1(s). \end{aligned} \quad (41)$$

s_0 and s_1 are the solutions to

$$\begin{cases} \sum_{\ell=0}^{n_t-1} \frac{(n_t-1)!}{\ell! s_0^{n_t-\ell}} = 1 + \frac{P}{n_t} s_0, \\ \sum_{\ell=0}^{n_t-1} \frac{(n_t-1)!}{\ell! s_1^{n_t-\ell}} = 1, \end{cases} \quad (42)$$

respectively.

Proof: Based on Theorem 2, transmitting each of the code layers on all available antennas and performing equal power allocation is optimum. As showed in [13], the maximum continuous-layer expected-rate of fading channels with general distribution is given by

$$\mathcal{R}_c^m = \max_{I(s)} \int_0^\infty \overline{F}_a(s) \frac{-sI'(s)}{1 + sI(s)} ds. \quad (43)$$

Noting $\overline{F}_a(s) = \frac{\Gamma(n_t, s)}{\Gamma(n_t)} = e^{-s} \sum_{\ell=0}^{n_t-1} \frac{s^\ell}{\ell!}$, we have

$$\mathcal{R}_c^m = \max_{I(s)} \int_0^\infty \frac{-s e^{-s} I'(s)}{1 + sI(s)} \sum_{\ell=0}^{n_t-1} \frac{s^\ell}{\ell!} ds. \quad (44)$$

The optimization solution to Eq. (44) with respect to $I(s)$ under the total power constraint $\frac{P}{n_t}$ at each antenna is found using variation methods [30]. By solving the corresponding Euler equation [30], we come up with the final solution as follows,

$$\mathcal{R}_c^m = \int_{s_0}^{s_1} e^{-s} \left(\frac{n_t + 1}{s} - 1 \right) \sum_{\ell=0}^{n_t-1} \frac{s^\ell}{\ell!} ds, \quad (45)$$

where boundaries s_0 and s_1 are the solutions to $\sum_{\ell=0}^{n_t-1} \frac{(n_t-1)!}{\ell! s_0^{n_t-\ell}} = 1 + \frac{P}{n_t} s_0$ and $\sum_{\ell=0}^{n_t-1} \frac{(n_t-1)!}{\ell! s_1^{n_t-\ell}} = 1$, respectively. The indefinite integral (antiderivative) of Eq. (45) is given by Eq. (41) (the derivation steps are deferred to appendix B). Applying the integration limits completes the proof. ■

Remark 4 *By substituting $n_t = 1$ in Proposition 4, the maximum continuous-layer expected-rate of the SISO channel is*

$$\mathcal{R}_c^m = 2E_1 \left(\frac{2}{1 + \sqrt{1 + 4P}} \right) - 2E_1(1) - e^{\frac{-2}{1 + \sqrt{1 + 4P}}} + e^{-1}. \quad (46)$$

As pointed out earlier, one can model a point-to-point block Rayleigh fading channel with an equivalent broadcast channel. According to the degradedness of the equivalent SISO broadcast channel, and the optimality of superposition (multi-layer) coding for such channels [27], the maximum continuous-layer expected-rate of the SISO channel, i.e., Eq. (46), represents its maximum average achievable rate [13].

Remark 5 Since the equivalent broadcast channel of the MISO channel is not degraded, its maximum continuous-layer expected-rate is not the maximum average achievable rate of the channel. For example, in asymptotically low SNR regime, the multiple-access scheme provides a higher average achievable rate in the MISO channel. In the multiple-access scheme, the antennas send independent messages, and the receiver decodes as much as it can.

Remark 6 Similar to remark 3, one can conclude that for $0 < s_i^o < 1$, $\forall i$, the maximum expected-rate of the MISO channel with uninformed transmitter is a Schur-concave function of the channel covariance matrix, that is channel correlation reduces the maximum expected-rate.

V. MAXIMUM THROUGHPUT IN MIMO CHANNELS

The throughput maximization problem in the MIMO channel is less tractable than that corresponding to the MISO channel.

Since in the Gaussian MIMO channel, in the sense of the outage probability, the optimum eigenvectors of the transmit covariance matrix always correspond to the eigenvectors of the channel correlation matrix [29], one can restrict the transmit covariance matrix to be diagonal in the problem of interest.

Recall from Section II-B, in an $n_t \times n_r$ MIMO channel, the PDF of the instantaneous mutual information in Eq. (3) does not lend itself to a closed form expression. In order to analyze the throughput, it is necessary to characterize this PDF. There are some approximations for the PDF of the instantaneous mutual information in literature, e.g., approximations on the distribution of the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ in MIMO channels with asymptotically large number of antennas at both the transmitter and receiver sides [31], [32].

In a MIMO channel with $\mathbf{Q} = \frac{P}{n_t} \mathbf{I}_{n_t}$, the PDF of the instantaneous mutual information can be well approximated by the Gaussian distribution with the same mean and variance [19], [20], i.e.,

$$\mathcal{I} \sim \mathcal{N}(\mu(n_t, n_r), \sigma^2(n_t, n_r)), \quad (47)$$

where

$$\begin{cases} \mu(n_t, n_r) = \mathbb{E}(\mathcal{I}), \\ \sigma^2(n_t, n_r) = \text{Var}(\mathcal{I}). \end{cases} \quad (48)$$

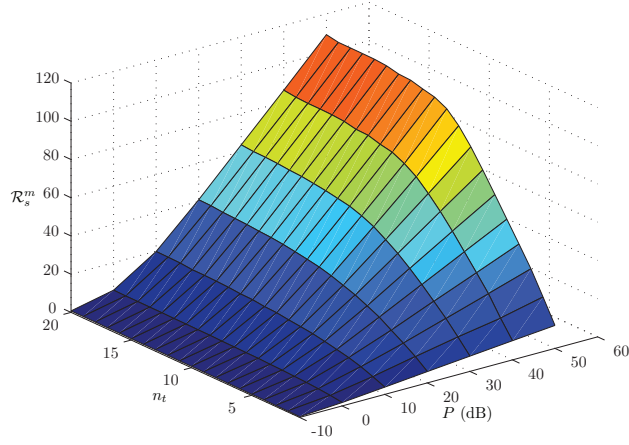


Fig. 1. The maximum throughput (in nats) in a MIMO channel with 10 receive antennas ($n_r = 10$).

Note that $\mu(n_t, n_r)$ equals the ergodic capacity of an $n_t \times n_r$ MIMO channel, which is a strictly increasing function with respect to n_t and n_r [6]. This Gaussian distribution approximation allows the throughput maximization to be expressed as

$$\begin{aligned} \mathcal{R}_s^m &= \max_R \Pr \{ \mathcal{I} \geq R \} R \\ &= \max_R \mathcal{Q} \left(\frac{R - \mu(n_t, n_r)}{\sigma(n_t, n_r)} \right) R. \end{aligned} \quad (49)$$

With $z = \frac{R - \mu(n_t, n_r)}{\sigma(n_t, n_r)}$, Eq. (49) leads to

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z) (\sigma(n_t, n_r)z + \mu(n_t, n_r)) \quad (50)$$

$$= \mathcal{Q}(z^o) (\sigma(n_t, n_r)z^o + \mu(n_t, n_r)), \quad (51)$$

where z^o is the solution to

$$-\frac{1}{\sqrt{2\pi}} e^{-\frac{z^o{}^2}{2}} (\sigma(n_t, n_r)z^o + \mu(n_t, n_r)) + \sigma(n_t, n_r) \mathcal{Q}(z^o) = 0. \quad (52)$$

Since the existing approximations for the PDF of the instantaneous mutual information in the MIMO channel are not tractable enough to analyze the maximum throughput in general case, four asymptotic cases are investigated. In all four cases, it is shown that the optimum transmit strategy is to use all available antennas. It seems reasonable to conjecture that the above statement holds with the general MIMO channel. To test the claim, Fig. 1 shows the maximum throughput in a MIMO channel with 10 receive antennas. Note that the number of transmit antennas varies from 1 to 20 and the total power P sweeps the range of -10 dB to 50 dB.

A. Asymptotically Low SNR Regime

For small SNR values, the eigenvalues of $\mathbf{Q}\mathbf{H}^\dagger\mathbf{H}$ are small enough to approximate the following,

$$\prod_{\ell=1}^{n_t} (1 + \text{eig}_{\ell}(\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H})) \approx 1 + \sum_{\ell=1}^{n_t} \text{eig}_{\ell}(\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}). \quad (53)$$

Therefore, the instantaneous mutual information of Eq. (3) can be approximated by

$$\begin{aligned} \mathcal{I} &= \ln \det (\mathbf{I}_{n_t} + \mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}) \\ &= \ln \prod_{\ell=1}^{n_t} (1 + \text{eig}_{\ell}(\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H})) \\ &\approx \ln \left(1 + \sum_{\ell=1}^{n_t} \text{eig}_{\ell}(\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}) \right). \end{aligned} \quad (54)$$

Using Eq. (54), we can prove the following proposition on the optimum transmit covariance matrix which maximizes the throughput in the asymptotically low SNR regime MIMO channel.

Proposition 5 *The optimum transmit strategy maximizing the throughput in the asymptotically low SNR regime MIMO channel is transmitting independent signals and performing equal power allocation across all available antennas. The maximum throughput is*

$$\mathcal{R}_s^m = \max_{0 < s < n_r} \frac{\Gamma(n_t n_r, n_t s)}{(n_t n_r - 1)!} \ln(1 + Ps). \quad (55)$$

Proof: Let $\delta_{\ell}P$ denote the allocated power to the ℓ 'th antenna subject to $\sum_{\ell=1}^{n_t} \delta_{\ell} = 1$. From Eq. (54), the instantaneous mutual information for low SNR values can be expressed as,

$$\begin{aligned} \mathcal{I} &\approx \ln \left(1 + \sum_{\ell=1}^{n_t} \text{eig}_{\ell}(\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}) \right) \\ &= \ln (1 + \text{tr}(\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H})) \\ &= \ln \left(1 + P \sum_{\ell=1}^{n_t} \sum_{k=1}^{n_r} \delta_{\ell} |h_{\ell,k}|^2 \right). \end{aligned} \quad (56)$$

Equation (56) corresponds to the instantaneous mutual information in the MISO channel. Therefore, the optimum transmit strategy minimizing the outage probability in the asymptotically low SNR regime MIMO channel is to transmit independent signals and perform equal power allocation across a fraction of available antennas.

Assume that the transmitter has allocated equal power to l_t out of n_t transmit antennas. the maximum throughput is given by

$$\mathcal{R}_s^m = \max_s \frac{\Gamma(l_t n_r, l_t s)}{(l_t n_r - 1)!} \ln(1 + Ps). \quad (57)$$

With $\hat{s} = \frac{s}{n_r}$, Eq. (57) leads to

$$\mathcal{R}_s^m = \max_{\hat{s}} \frac{\Gamma(l_t n_r, l_t n_r \hat{s})}{(l_t n_r - 1)!} \ln(1 + P n_r \hat{s}). \quad (58)$$

Equation (58) corresponds to the maximum throughput expression of the MISO channel, i.e., Eq. (16), with $l_t n_r$ transmit antennas and total power $P n_r$. According to Theorem 1, the optimum transmit strategy is to use all available antennas and $0 < \hat{s} < 1$, and equivalently $0 < s < n_r$. ■

In the same direction, the finite-layer expected-rate is given by Corollary 2.

Corollary 2 *The optimum transmit strategy maximizing the K -layer expected-rate of the asymptotically low SNR regime MIMO channel is transmitting independent signals and performing equal power allocation across all available antennas in each code layer. The maximum throughput is*

$$\mathcal{R}_f^m = \max_{\substack{0 < s_i < n_r, P_i \\ \sum_{i=1}^K P_i = P}} \sum_{i=1}^K \frac{\Gamma(n_t n_r, n_t s)}{(n_t n_r - 1)!} \ln \left(1 + \frac{P_i s_i}{1 + \sum_{j=i+1}^K P_j s_j} \right). \quad (59)$$

Proof: At the i 'th layer, let $\delta_\ell P_i$ and $\eta_\ell I_i$ denote the allocated power and upper-layers power at the ℓ 'th antenna subject to $\sum_{\ell=1}^{n_t} \delta_\ell = \sum_{\ell=1}^{n_t} \eta_\ell = 1$, and $I_i = \sum_{j=i+1}^K P_j$. Following the same steps in Eq. (56), the i 'th layer instantaneous mutual information can be approximated by

$$\mathcal{I}_i \approx \ln \left(1 + \frac{P_i \sum_{\ell=1}^{n_t} \sum_{k=1}^{n_r} \delta_\ell |h_{\ell,k}|^2}{1 + I_i \sum_{\ell=1}^{n_t} \sum_{k=1}^{n_r} \eta_\ell |h_{\ell,k}|^2} \right). \quad (60)$$

Equation (60) corresponds to the instantaneous mutual information of the multi-layer MISO channel in Section IV-A. The proof is completed by following the steps in the proof of Theorem 2 and Proposition 5. ■

Corresponding to Proposition 4, we have the following corollary for continuous-layer coding in the low SNR MIMO channels.

Corollary 3 *The maximum continuous-layer expected-rate in the asymptotically low SNR regime MIMO channel is given by*

$$\mathcal{R}_c^m = \mathcal{R}(s_1) - \mathcal{R}(s_0), \quad (61)$$

where,

$$\begin{aligned} \mathcal{R}(s) = e^{-s} \sum_{\ell=1}^{n_t n_r - 1} \frac{1}{\ell!} \left(s^\ell - (n_t n_r + 1 - \ell)(\ell - 1)! \sum_{k=0}^{\ell-1} \frac{s^k}{k!} \right) \\ + e^{-s} - (n_t n_r + 1) E_1(s). \end{aligned} \quad (62)$$

s_0 and s_1 are the solutions to

$$\begin{cases} \sum_{\ell=0}^{n_t n_r - 1} \frac{(n_t n_r - 1)!}{\ell! s_0^{n_t n_r - \ell}} = 1 + \frac{P}{n_t} s_0, \\ \sum_{\ell=0}^{n_t n_r - 1} \frac{(n_t n_r - 1)!}{\ell! s_1^{n_t n_r - \ell}} = 1, \end{cases} \quad (63)$$

respectively.

Remark 7 Analogous to the MISO channel, in the asymptotically low SNR regime MIMO channel with uninformed transmitter, channel correlation decreases the maximum throughput and maximum expected-rate.

B. Asymptotically High SNR Regime

For large SNR values, we take advantages of Wishart distribution properties. In order to enhance the lucidity of this section, let us define $p \triangleq \min \{n_t, n_r\}$, $n \triangleq \max \{n_t, n_r\}$, and

$$\mathbf{W} = \begin{cases} \mathbf{H}^\dagger \mathbf{H} & n_t \leq n_r, \\ \mathbf{H} \mathbf{H}^\dagger & n_t > n_r. \end{cases} \quad (64)$$

Matrix \mathbf{W} has a central complex p -variate Wishart distribution with scale matrix \mathbf{I}_{n_t} and n degrees of freedom [33]–[35].

Theorem 3 yields the maximum throughput in the asymptotically high SNR regime MIMO channel by obtaining the optimum transmit covariance matrix \mathbf{Q}^o .

Theorem 3 The optimum transmit strategy maximizing the throughput in the asymptotically high SNR regime MIMO channel is sending independent signals and performing equal power allocation across all available antennas. The maximum throughput is

$$\mathcal{R}_s^m = \max_s \bar{F}_a \left(\frac{n_t^p s}{P^{p-1}} \right) \ln(1 + Ps) \quad (65)$$

$$= \max_z \mathcal{Q}(z) \left(z \sqrt{\frac{\pi^2}{6} p - \sum_{k=0}^{p-1} \sum_{\ell=1}^{n-k-1} \frac{1}{\ell^2}} \right. \\ \left. + p \left(F(1) + \ln \left(\frac{P}{n_t} \right) \right) + \sum_{k=0}^{p-1} \sum_{\ell=1}^{n-k-1} \frac{1}{\ell} \right), \quad (66)$$

where $-F(1) \approx 0.577215$ is the Euler-Mascheroni constant, $a \triangleq \prod_{\ell=1}^p a_{\ell,\ell}^2$, and $a_{\ell,\ell}^2, \forall \ell$ are independent gamma-distributed with scale 1 and shape $n - \ell + 1$, i.e., $f_{a_{\ell,\ell}^2}(x) = \frac{\Gamma(n-\ell+1, x)}{(n-\ell)!}$.

Proof: Again, we first assume that l_t out of n_t antennas are active. Then, we shall see that the optimum solution is $l_t^o = n_t$. Define the index set $Z(\mathbf{Q}) \triangleq \{\ell : q_{\ell,\ell} = 0\}$. Denote by \mathbf{Q}_{l_t} the matrix obtained from \mathbf{Q} by eliminating of all the ℓ 'th rows and columns with $\ell \in Z(\mathbf{Q})$. Clearly, \mathbf{Q}_{l_t} has full rank. We divide the proof into two parts: Part i) $l_t \leq n_r$, Part ii) $l_t \geq n_r$. We wish to show that in both cases, the throughput is a strictly increasing function with respect to l_t .

Part i):

In high SNR regime, the eigenvalues of $\mathbf{Q}_{l_t}\mathbf{H}^\dagger\mathbf{H}$ are large. The instantaneous mutual information can be well approximated by

$$\begin{aligned}
\mathcal{I} &= \ln \det (\mathbf{I}_{l_t} + \mathbf{Q}\mathbf{H}^\dagger\mathbf{H}) \\
&= \ln \prod_{\ell=1}^{l_t} (1 + \text{eig}_\ell (\mathbf{Q}\mathbf{H}^\dagger\mathbf{H})) \\
&\approx \ln \prod_{\ell=1}^{l_t} (\text{eig}_\ell (\mathbf{Q}_{l_t}\mathbf{H}^\dagger\mathbf{H})) \\
&= \ln \det (\mathbf{Q}_{l_t}\mathbf{H}^\dagger\mathbf{H}) \\
&= \ln \det \mathbf{Q}_{l_t} + \ln \det (\mathbf{H}^\dagger\mathbf{H}) \\
&= \ln \det \mathbf{Q}_{l_t} + \ln \det \mathbf{W}.
\end{aligned} \tag{67}$$

Clearly, the CDF of $\ln \det \mathbf{W}$ decreases by the use of more antennas. We shall now show that $\ln \det \mathbf{Q}_{l_t}$ and thereby, \mathcal{I} increases with the number of active antennas. It is straight forward to verify that the solution to the maximization problem $\max \det \mathbf{Q}_{l_t}$ subject to $\text{tr}(\mathbf{Q}_{l_t}) = P$ over diagonal matrices is $\mathbf{Q}_{l_t} = \frac{P}{l_t}\mathbf{I}_{l_t}$. Therefore, Eq. (67) is simplified as follows

$$\mathcal{I} \approx l_t \ln \left(\frac{P}{l_t} \right) + \ln \det \mathbf{W}. \tag{68}$$

For $P > el_t$,

$$\frac{\partial \mathcal{I}}{\partial l_t} = \ln \left(\frac{P}{l_t} \right) - 1 > 0. \tag{69}$$

As a result, in high SNR regime, the instantaneous mutual information \mathcal{I} strict monotonic increasing with respect to the number of transmit antennas.

Part ii):

In this case, we approximate the instantaneous mutual information as follows.

$$\begin{aligned}
\mathcal{I} &= \ln \det (\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger) \\
&= \ln \prod_{\ell=1}^{n_r} (1 + \text{eig}_\ell (\mathbf{H}\mathbf{Q}\mathbf{H}^\dagger)) \\
&\approx \ln \prod_{\ell=1}^{n_r} (\text{eig}_\ell (\mathbf{H}\mathbf{Q}_{l_t}\mathbf{H}^\dagger)) \\
&= \ln \det (\mathbf{H}\mathbf{Q}_{l_t}\mathbf{H}^\dagger).
\end{aligned} \tag{70}$$

In this case, let us assume that the transmitter performs equal power allocation. Therefore,

$$\begin{aligned}
\mathcal{I} &\approx n_r \ln \left(\frac{P}{l_t} \right) + \ln \det (\mathbf{H}\mathbf{H}^\dagger) \\
&= n_r \ln \left(\frac{P}{l_t} \right) + \ln \det \mathbf{W}.
\end{aligned} \tag{71}$$

In the following, we shall establish that the maximum throughput of the channel is strictly increasing with respect to l_t . From the maximization problem of Eq. (50), the maximum throughput can be equivalently expressed as

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z) (\sigma(l_t, n_r)z + \mu(l_t, n_r)), \quad (72)$$

with

$$\mu(l_t, n_r) = \mathbb{E} (\ln \det \mathbf{W}) + p \ln \left(\frac{P}{l_t} \right), \quad (73)$$

$$\sigma^2(l_t, n_r) = \text{Var} (\ln \det \mathbf{W}). \quad (74)$$

A central complex Wishart-distributed matrix \mathbf{W} satisfies [23]

$$\mathbb{E} (\ln \det \mathbf{W}) = \sum_{k=0}^{p-1} F(n-k), \quad (75)$$

$$\text{Var} (\ln \det \mathbf{W}) = \sum_{k=0}^{p-1} F'(n-k). \quad (76)$$

For natural arguments, the Eüler's digamma function and its derivative, i.e., $F(m)$ and $F'(m)$, can be expressed as

$$F(m) = F(1) + \sum_{\ell=1}^{m-1} \frac{1}{\ell}, \quad (77)$$

$$F'(m) = \frac{\pi^2}{6} - \sum_{\ell=1}^{m-1} \frac{1}{\ell^2}, \quad (78)$$

with $-F(1) = -\Gamma'(1) = \lim_{m \rightarrow \infty} (\sum_{\ell=1}^m \frac{1}{\ell} - \ln(m)) \approx 0.577215$ the Eüler-Mascheroni constant. Inserting Eq. (78) into Eq. (76) and then into Eq. (74) to obtain

$$\sigma^2(l_t, n_r) = \frac{\pi^2}{6} n_r - \sum_{k=0}^{n_r-1} \sum_{\ell=1}^{l_t-k-1} \frac{1}{\ell^2}, \quad (79)$$

we see that $\sigma^2(l_t, n_r)$ is a monotonically decreasing function with respect to l_t . Whereas $\mu(l_t, n_r)$ is a strictly increasing function with respect to both l_t and n_r as it represents the ergodic capacity of the high SNR $l_t \times n_r$ MIMO channel. On the other hand, $\sigma^2(l_t, n_r) = \sum_{k=0}^{p-1} F'(n-k)$ is a monotonically increasing function with respect to n_r , because of the Basel problem, i.e., $\lim_{m \rightarrow \infty} \sum_{\ell=1}^m \frac{1}{\ell^2} = \frac{\pi^2}{6}$, which verifies that $F'(m) \geq 0$.

As the \mathcal{Q} -function is upper-bounded by the Chernoff bound, i.e., $\mathcal{Q}(z) \leq \frac{1}{2} e^{-\frac{z^2}{2}}$, $z \geq 0$, we have for $z \geq 0$,

$$\begin{aligned} & -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} (\sigma(l_t, n_r)z + \mu(l_t, n_r)) + \sigma(l_t, n_r) \mathcal{Q}(z) \\ & \leq -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma(l_t, n_r) \left(z + \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} \right) \stackrel{(a)}{<} 0, \end{aligned} \quad (80)$$

where (a) follows the fact that $z \geq 0$ and $\frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} > 0$ as P and thereby $\mu(l_t, n_r)$ is large. From Eqs. (52) and (80), one immediately finds that $z^o < 0$. Recall from Eq. (51), the maximum throughput is a strictly increasing function with respect to l_t because \mathcal{R}_s^m is a strictly increasing function with respect to $\mu(l_t, n_r)$, a monotonically decreasing function with respect to $\sigma(l_t, n_r)$, and $z^o < 0$.

Thus, in both parts, i.e., $l_t \leq n_r$ and $l_t \geq n_r$, \mathcal{R}_s^m is a strictly increasing function with respect to l_t . We conclude that in the asymptotically high SNR regime MIMO channel, the maximum throughput is a strictly increasing function with respect to the number of active transmit antennas, and hence, $l_t^o = n_t$.

Performing Bartlett decomposition [36], we get $\mathbf{W} = \mathbf{A}\mathbf{A}^\dagger$, where \mathbf{A} is a square lower triangular matrix (left triangular matrix) in the form of

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p,1} & a_{p,2} & a_{p,3} & \cdots & a_{p,p} \end{bmatrix}, \quad (81)$$

where $a_{\ell,k} \sim \mathcal{CN}(0, 1)$, $\ell \neq k$, and $a_{\ell,\ell}^2, \forall \ell$ are independent gamma-distributed with scale 1 and shape $n - \ell + 1$. Clearly, $\det \mathbf{W} = \det \mathbf{A} \times \det \mathbf{A}^\dagger = \prod_{\ell=1}^p a_{\ell,\ell}^2$.

Therefore, the maximum throughput is

$$\begin{aligned} \mathcal{R}_s^m &= \max_s \Pr \left\{ \det \left(\frac{P}{n_t} \mathbf{W} \right) \geq Ps \right\} \ln(1 + Ps) \\ &= \max_s \Pr \left\{ \det \mathbf{W} \geq \frac{n_t^p s}{P^{p-1}} \right\} \ln(1 + Ps) \\ &= \max_s \Pr \left\{ \prod_{\ell=1}^p a_{\ell,\ell}^2 \geq \frac{n_t^p s}{P^{p-1}} \right\} \ln(1 + Ps). \end{aligned} \quad (82)$$

From Eqs. (72) to (79), the throughput can also be written as

$$\begin{aligned} \mathcal{R}_s^m &= \max_z \mathcal{Q}(z) (\sigma(n_t, n_r) z + \mu(n_t, n_r)) \\ &= \max_z \mathcal{Q}(z) \left(z \sqrt{\sum_{\ell=0}^{p-1} F'(n - \ell)} \right. \\ &\quad \left. + p \ln \left(\frac{P}{n_t} \right) + \sum_{\ell=0}^{p-1} F(n - \ell) \right) \\ &= \max_z \mathcal{Q}(z) \left(z \sqrt{\frac{\pi^2}{6} p - \sum_{k=0}^{p-1} \sum_{\ell=1}^{n-k-1} \frac{1}{\ell^2}} \right. \\ &\quad \left. + p \left(F(1) + \ln \left(\frac{P}{n_t} \right) \right) + \sum_{k=0}^{p-1} \sum_{\ell=1}^{n-k-1} \frac{1}{\ell} \right). \end{aligned} \quad (83)$$

Remark 8 Since in asymptotically high SNR regime, the outage probability is Schur-convex with respect to the channel covariance matrix [10], the maximum throughput is a Schur-concave function of the channel covariance matrix, i.e., channel correlation decreases the maximum throughput.

C. Asymptotically Large Number of Antennas

Here, two asymptotic results for large number of transmit antennas and large number of receive antennas are presented. As pointed out earlier, we can restrict our attention to diagonal transmit covariance matrices. To prove by contradiction, first we assume that the optimum transmit covariance matrix is $\mathbf{Q}^o = \frac{P}{l_t} \mathbf{I}_{l_t}$; next, we shall show that the maximum throughput increases with the number of transmit antennas and hence, $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$. Finally, we formulate the maximum throughput.

In following, Theorems 4 and 5 yield the maximum throughput of asymptotically large number of transmit antennas and asymptotically large number of receive antennas, respectively. In the proof of both theorems, we use the results presented by Hochwald, Marzetta, and Tarokh [20] which provide us with approximations for mean and variance of the instantaneous mutual information in the large number of transmit antennas and large number of receive antennas asymptotes.

Theorem 4 In the MIMO channel with asymptotically large number of transmit antennas, the optimum transmit covariance matrix which maximizes the throughput is $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$. The maximum throughput of the channel is given by

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z) \left(\sqrt{\frac{n_r}{n_t}} \frac{P}{\sqrt{1+P^2}} z + n_r \ln(1+P) \right). \quad (84)$$

Proof: According to the results provided in [20], we have

$$\begin{cases} \lim_{n_t \rightarrow \infty} \mu(l_t, n_r) = n_r \ln(1+P), \\ \lim_{n_t \rightarrow \infty} \sigma^2(l_t, n_r) = \frac{n_r P^2}{l_t(1+P^2)}. \end{cases} \quad (85)$$

From Eq. (85) and noting the \mathcal{Q} -function's Chernoff bound, i.e., $\mathcal{Q}(z) \leq \frac{1}{2} e^{-\frac{z^2}{2}}$, $z \geq 0$, we have for $z \geq 0$,

$$\begin{aligned} & -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} (\sigma(l_t, n_r) z + \mu(l_t, n_r)) + \sigma(l_t, n_r) \mathcal{Q}(z) \\ & \leq -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma(l_t, n_r) \left(z + \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} \right) \stackrel{(a)}{<} 0, \end{aligned} \quad (86)$$

where (a) comes from the fact that for $z \geq 0$,

$$\begin{aligned} z + \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} &\geq \frac{\mu(l_t, n_r)}{\sigma(l_t, n_r)} - \sqrt{\frac{\pi}{2}} \\ &= \sqrt{n_r l_t} \sqrt{1 + \frac{1}{P^2} \ln(1+P)} - \sqrt{\frac{\pi}{2}} \xrightarrow{l_t \rightarrow \infty} 0. \end{aligned} \quad (87)$$

Comparing Eqs. (52) and (86), we have $z^o < 0$. Since $\mu(l_t, n_r)$ does not depend on l_t , $\sigma(l_t, n_r)$ is a strictly decreasing functions with respect to l_t , and $z^o < 0$, one can conclude that $\mathcal{R}_s^m = \mathcal{Q}(z^o) (\sigma(l_t, n_r) z^o + \mu(l_t, n_r))$ is a strictly increasing function with respect to l_t . Thus, $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$. ■

Theorem 5 *In the MIMO channel with asymptotically large number of receive antennas, the optimum transmit covariance matrix which maximizes the throughput is $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$. The maximum throughput of the channel is given by*

$$\mathcal{R}_s^m = \max_z \mathcal{Q}(z) \left(\sqrt{\frac{n_t}{n_r}} z + n_t \ln \left(1 + \frac{n_r}{n_t} P \right) \right). \quad (88)$$

Proof: As the number of receive antennas goes to infinity, the mean and variance of the channel mutual information obey [20]

$$\begin{cases} \lim_{n_r \rightarrow \infty} \mu(l_t, n_r) = l_t \ln \left(1 + \frac{n_r}{l_t} P \right), \\ \lim_{n_r \rightarrow \infty} \sigma^2(l_t, n_r) = \frac{l_t}{n_r}. \end{cases} \quad (89)$$

From Eqs. (50) and (85), the maximum throughput is

$$\begin{aligned} \mathcal{R}_s^m &= \max_z \mathcal{Q}(z) \left(\sqrt{\frac{l_t}{n_r}} z + l_t \ln \left(1 + \frac{n_r}{l_t} P \right) \right) \\ &\stackrel{(a)}{\geq} \mathcal{Q}(-\sqrt{n_r}) \left(-\sqrt{l_t} + l_t \ln \left(1 + \frac{n_r}{l_t} P \right) \right) \\ &\stackrel{(b)}{>} \mathcal{Q}(-\sqrt{n_r}) \left(-\ln \left(1 + \frac{n_r P}{l_t - 1} \right) \right. \\ &\quad \left. - l_t \ln \left(1 - \frac{1}{l_t} \right) + l_t \ln \left(1 + \frac{n_r}{l_t} P \right) \right) \\ &\stackrel{(c)}{>} \mathcal{Q}(-\sqrt{n_r}) \left((l_t - 1) \ln \left(1 + \frac{n_r}{l_t - 1} P \right) \right) \\ &\stackrel{(d)}{\geq} \left(1 - \frac{1}{2} e^{-\frac{n_r}{2}} \right) \left((l_t - 1) \ln \left(1 + \frac{n_r}{l_t - 1} P \right) \right) \\ &\stackrel{(e)}{\xrightarrow{n_r \rightarrow \infty}} (l_t - 1) \ln \left(1 + \frac{n_r}{l_t - 1} P \right) \\ &\stackrel{(f)}{\geq} \max_z \mathcal{Q}(z) \left(\sqrt{\frac{l_t - 1}{n_r}} z + (l_t - 1) \ln \left(1 + \frac{n_r}{l_t - 1} P \right) \right), \end{aligned} \quad (90)$$

where (a) follows from choosing $z = -\sqrt{n_r}$ instead of its optimum value, (b) follows from $\sqrt{l_t} + l_t \ln\left(\frac{l_t}{l_t-1}\right) < \ln\left(1 + \frac{n_r P}{l_t-1}\right)$ for large values of n_r , (c) follows from algebraic simplifications, (d) follows from the \mathcal{Q} -function's Chernoff bound, (e) follows from $\lim_{n_r \rightarrow \infty} e^{-\frac{n_r}{2}} \ln\left(1 + \frac{n_r}{l_t-1} P\right) = 0$, and (f) follows from the fact that the maximum throughput is always less than or equal to the ergodic capacity based on Proposition 1.

Equation (90) proves that \mathcal{R}_s^m is a strictly increasing function with respect to l_t , and hence, $\mathbf{Q}^o = \frac{P}{n_t} \mathbf{I}_{n_t}$. ■

VI. TWO-TRANSMITTER DISTRIBUTED ANTENNA SYSTEMS

There has been some research in assumption of perfect cooperation between base stations, and consequently treat them as distributed antennas of one base station [37]. Here, we investigate a block Rayleigh fading system wherein two uninformed single-antenna transmitters want to transmit a common message to a single-antenna receiver. Let h_1 and h_2 denote the fading coefficients of the first transmitter-receiver link and second transmitter-receiver link, respectively. We assume that h_1 and h_2 are independent i.i.d. complex Gaussian random variables, each with zero-mean and equal variance real and imaginary parts ($h_1, h_2 \sim \mathcal{CN}(0, 1)$). We also assume that h_1 and h_2 are constant during two consecutive transmission blocks.

We propose a practical distributed algorithm that provides all instantaneous mutual information distributions which are achievable by treating the transmitters as antennas of one composed element. Theorem 6 proves that the outage probability in a MISO channel with two transmit antennas is also achievable in this channel.

Theorem 6 *The outage probability in a MISO channel with two transmit antennas and total power constraint P is achievable in a distributed antenna system with two single-antenna transmitters and one single-antenna receiver, where the total power constraint at each transmitter is $\frac{P}{2}$.*

Proof: To prove the statement, first, a general expression for the outage probability in a 2×1 MISO channel is derived. Afterwards, we shall show that this expression is achievable in the two-transmitter distributed antenna system.

In the 2×1 MISO channel, the outage probability for transmission rate R is expressed as

$$\mathcal{P}_{\text{out}} = \Pr \left\{ \ln \left(1 + \vec{h} \mathbf{Q} \vec{h}^\dagger \right) < R \right\}, \quad (91)$$

where \mathbf{Q} is the transmit covariance matrix. Since \mathbf{Q} is non-negative definite, one can write it as $\mathbf{Q} = \mathbf{U} \mathbf{D} \mathbf{U}^\dagger$, where \mathbf{D} is diagonal and \mathbf{U} is unitary. As h_1 and h_2 are independent complex Gaussian random

variables, each with independent zero-mean and equal variance real and imaginary parts, the distribution of $\vec{h}\mathbf{U}$ is the same as that of \vec{h} [6]. Thus, Eq. (91) is simplified to

$$\begin{aligned}\mathcal{P}_{\text{out}} &= \Pr \left\{ \ln \left(1 + \left(\vec{h}\mathbf{U} \right) \mathbf{D} \left(\vec{h}\mathbf{U} \right)^\dagger \right) < R \right\} \\ &= \Pr \left\{ \ln \left(1 + \vec{h}\mathbf{D}\vec{h}^\dagger \right) < R \right\}.\end{aligned}\quad (92)$$

Since $\mathbf{U}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is unitary, the distribution of $\vec{h}\mathbf{U}_0$ is the same as that of \vec{h} . Inserting into Eq. (92) yields

$$\begin{aligned}\mathcal{P}_{\text{out}} &= \Pr \left\{ \ln \left(1 + \left(\vec{h}\mathbf{U}_0 \right) \mathbf{D} \left(\vec{h}\mathbf{U}_0 \right)^\dagger \right) < R \right\} \\ &= \Pr \left\{ \ln \left(1 + \vec{h} (\mathbf{U}_0 \mathbf{D} \mathbf{U}_0) \vec{h}^\dagger \right) < R \right\}.\end{aligned}\quad (93)$$

Since $\text{tr}(\mathbf{Q}) = \text{tr}(\mathbf{D})$, the total power constraint can be written as $\text{tr}(\mathbf{D}) \leq P$. Without loss of generality, let us define $\mathbf{D} \triangleq P \begin{bmatrix} \delta & 0 \\ 0 & \bar{\delta} \end{bmatrix}$, where $0 \leq \delta \leq 1$ and $\bar{\delta} = 1 - \delta$. Inserting into Eq. (93) yields

$$\mathcal{P}_{\text{out}} = \Pr \left\{ \ln \left(1 + \vec{h} \frac{P}{2} \begin{bmatrix} 1 & 2\delta - 1 \\ 2\delta - 1 & 1 \end{bmatrix} \vec{h}^\dagger \right) < R \right\}.\quad (94)$$

Defining $\rho \triangleq 2\delta - 1$, we get

$$\begin{aligned}\mathcal{P}_{\text{out}} &= \Pr \left\{ \ln \left(1 + \vec{h} \frac{P}{2} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \vec{h}^\dagger \right) < R \right\} \\ &= \Pr \left\{ \ln \left(1 + (|h_1|^2 + |h_2|^2 + 2\rho \Re(h_1 h_2^*)) \frac{P}{2} \right) < R \right\}.\end{aligned}\quad (95)$$

Note that as $0 \leq \delta \leq 1$, we have $-1 \leq \rho \leq 1$.

We shall now show that the outage probability in Eq. (95) is achievable in the two-transmitter distributed antenna system with power constraint $\frac{P}{2}$ at each transmitter.

The transmission strategy in two consecutive time slots is as follows. In time slot t , the first (resp. second) transmitter sends $X(t)$ (resp. $\rho X(t) + \sqrt{(1 - \rho^2)}X(t + 1)$). In time slot $t + 1$, the first (resp. second) transmitter sends $-X^*(t + 1)$ (resp. $-\rho X^*(t + 1) + \sqrt{(1 - \rho^2)}X^*(t)$). Assuming $\mathbb{E}(|X|^2) = \frac{P}{2}$, the power consumption per time slot in each transmitter is $\frac{P}{2}$.

The received signal at the receiver is

$$\begin{aligned}Y(t) &= h_1 X(t) + h_2 \left(\rho X(t) \right. \\ &\quad \left. + \sqrt{(1 - \rho^2)} X(t + 1) \right) + Z(t),\end{aligned}\quad (96)$$

$$\begin{aligned}Y(t + 1) &= -h_1 X^*(t + 1) + h_2 \left(-\rho X^*(t + 1) \right. \\ &\quad \left. + \sqrt{(1 - \rho^2)} X^*(t) \right) + Z(t + 1).\end{aligned}\quad (97)$$

In matrix form,

$$\begin{bmatrix} Y(t) \\ -Y(t+1)^* \end{bmatrix} = \mathbf{G} \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} + \begin{bmatrix} Z(t) \\ -Z^*(t+1) \end{bmatrix}, \quad (98)$$

where

$$\mathbf{G} \triangleq \begin{bmatrix} h_1 + h_2 \rho & h_2 \sqrt{(1 - \rho^2)} \\ -h_2^* \sqrt{(1 - \rho^2)} & h_1^* + h_2^* \rho \end{bmatrix}. \quad (99)$$

By multiplying \mathbf{G}^\dagger to the both sides of Eq. (98), two parallel channels are separated as

$$\begin{aligned} \begin{bmatrix} \tilde{Y}(t) \\ \tilde{Y}(t+1) \end{bmatrix} &= \mathbf{G}^\dagger \begin{bmatrix} Y(t) \\ -Y^*(t+1) \end{bmatrix} \\ &= \left(|h_1 + h_2 \rho|^2 \right. \\ &\quad \left. + |h_2|^2 (1 - \rho^2) \right) \mathbf{I}_2 \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} \\ &\quad + \mathbf{G}^\dagger \begin{bmatrix} Z(t) \\ -Z^*(t+1) \end{bmatrix} \\ &= h \mathbf{I}_2 \begin{bmatrix} X(t) \\ X(t+1) \end{bmatrix} + \begin{bmatrix} \tilde{Z}(t) \\ \tilde{Z}(t+1) \end{bmatrix}, \end{aligned} \quad (100)$$

where $h \triangleq |h_1 + h_2 \rho|^2 + |h_2|^2 (1 - \rho^2)$, and $\tilde{Z}(t)$ and $\tilde{Z}(t+1)$ are independent zero mean complex Gaussian random variables with power equal to $\mathbb{E} \left(|\tilde{Z}|^2 \right) = h$. Thus, the received signal power to noise ratio at the receiver is

$$\begin{aligned} \frac{h^2 \frac{P}{2}}{\mathbb{E} \left(|\tilde{Z}|^2 \right)} &= (|h_1 + h_2 \rho|^2 + |h_2|^2 (1 - \rho^2)) \frac{P}{2} \\ &= (|h_1|^2 + |h_2|^2 + 2\rho \Re(h_1 h_2^*)) \frac{P}{2}. \end{aligned} \quad (101)$$

Therefore, the outage probability in the proposed scheme is given by

$$\mathcal{P}_{\text{out}} = \Pr \left\{ \ln \left(1 + (|h_1|^2 + |h_2|^2 + 2\rho \Re(h_1 h_2^*)) \frac{P}{2} \right) < R \right\}. \quad (102)$$

Equation (95) together with Eq. (102) shows that the outage probability in a 2×1 MISO channel is also achievable in the two-transmitter distributed antenna system. ■

Remark 9 To achieve the minimum outage probability in Theorem 6, the optimum solution to δ is either 1 or $\frac{1}{2}$, depending on R and P . Equivalently, in the two-transmitter distributed antennas, the optimum value of ρ is either 1 or 0.

Note that for $\rho = 0$, the proposed transmission scheme in the two-transmitter distributed antenna system is equivalent to the Alamouti code [38].

Remark 10 *Since the outage probability is the CDF of the instantaneous mutual information, one concludes that any achievable instantaneous mutual information distribution in the 2×1 MISO channel is also achievable in this two-transmitter distributed antenna system.*

Remark 11 *Based on Theorem 6, the maximum throughput in the two-transmitter distributed antenna system with total power constraint $\frac{P}{2}$ at each transmitter is the same as that of a 2×1 MISO channel with total power constraint P . By substituting $n_t = 2$ in Eq. (15), the maximum throughput is given by*

$$\mathcal{R}_s^m = \max_{0 < s < 1} (1 + 2s)e^{-2s} \ln(1 + Ps). \quad (103)$$

Remark 12 *In a similar approach, it can be shown that the maximum expected-rate as well as the ergodic capacity of this two-transmitter distributed antenna system and the 2×1 MISO channel are the same.*

Based on Theorem 6 and recall from Proposition 4 with $n_t = 2$, we come up with the following Corollary.

Corollary 4 *The maximum continuous-layer expected-rate of the distributed antenna system with two transmitters each with total power $\frac{P}{2}$ is*

$$\mathcal{R}_c^m = 3\mathbb{E}_1(s_0) + (1 - s_0)e^{-s_0} - 3\mathbb{E}_1(s_1) - (1 - s_1)e^{-s_1}, \quad (104)$$

where $s_1 = \frac{1+\sqrt{5}}{2}$, and $s_0 = \sqrt[3]{\sqrt{A^2 - B^3} + A} + \frac{B}{\sqrt[3]{\sqrt{A^2 - B^3} + A}} - \frac{2}{3P}$ with $A = \frac{1}{P} - \frac{2}{3P^2} - \frac{8}{27P^3}$ and $B = \frac{2}{3P} + \frac{4}{9P^2}$.

From Proposition 3, the ergodic capacity in this channel is

$$C_{\text{erg}} = 1 + \left(1 - \frac{2}{P}\right) e^{\frac{2}{P}} \mathbb{E}_1\left(\frac{2}{P}\right). \quad (105)$$

The maximum throughput, the maximum two-layer expected-rate, the maximum continuous-layer expected-rate, and the ergodic capacity in the two-transmitter distributed antenna system are depicted in Fig. 2.

VII. CONCLUSION

The throughput and expected-rate maximization of multiple-antenna channels are addressed in block Rayleigh fading environments, in which the transmitter does not access the CSI. It is established that, in order to achieve the maximum throughput, one has to transmit uncorrelated circularly symmetric zero mean equal power Gaussian signals from all the transmit antennas. This indeed yields the same transmit covariance matrix that achieves the ergodic capacity.

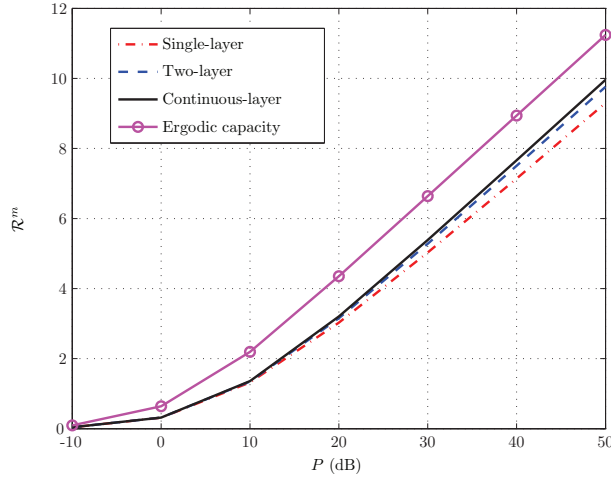


Fig. 2. The maximum throughput, the maximum two-layer expected-rate, the maximum continuous-layer expected-rate, and the ergodic capacity (all in *nats*) in the two-transmitter distributed antenna system.

In point-to-point uncorrelated MISO channels, in contrast to using a fraction of antennas which is optimum for outage capacity, the throughput is maximized by sending uncorrelated equal power signals on all transmit antennas. The maximum expected-rate is analyzed using multi-layer codes. It is proved that in each layer, sending uncorrelated signals with equal powers from all available antennas is optimum. The continuous-layer expected-rate of the channel is then derived in closed form.

The optimum transmit strategy maximizing the throughput is obtained for point-to-point uncorrelated MIMO channels. Since the PDF of the MIMO instantaneous mutual information is not tractable, four asymptotic cases are considered: low SNR regime, high SNR regime, large number of transmit antennas, and large number of receive antennas. In each case, the maximum throughput of the MIMO channel is derived.

Finally, a distributed antenna system with two single-antenna transmitters and one single-antenna receiver is investigated. It is proved that any achievable instantaneous mutual information distribution in the 2×1 MISO channel is also achievable in the two-transmitter distributed antenna system. Hence, both systems achieve the same maximum throughput and expected-rate.

APPENDIX A

PROOF OF PROPOSITION 3

The ergodic capacity of a $1 \times n_r$ SIMO channel is given by

$$C_{\text{erg}} = \int_0^\infty \frac{x^{n_r-1} e^{-x}}{(n_r-1)!} \ln(1+Px) dx. \quad (106)$$

Applying the integration by parts rule on Eq. (106) leads to

$$C_{\text{erg}} = \left[-e^{-x} \sum_{\ell=0}^{n_r-1} \frac{x^\ell}{\ell!} \ln(1+Px) \right]_0^\infty + \int_0^\infty e^{-x} \sum_{\ell=0}^{n_r-1} \frac{x^\ell}{\ell!} \frac{P}{1+Px} dx. \quad (107)$$

One can simply show that the first part on the right-hand-side in Eq. (107) is zero by repeatedly applying l'Hôpital's rule. With $t = 1 + Px$, Eq. (107) yields

$$C_{\text{erg}} = \int_1^\infty e^{-\frac{t-1}{P}} \sum_{\ell=0}^{n_r-1} \frac{1}{t\ell!} \left(\frac{t-1}{P} \right)^\ell dt. \quad (108)$$

From $(t-1)^\ell = \sum_{\iota=0}^\ell \binom{\ell}{\iota} t^\iota (-1)^{\ell-\iota}$, where $\binom{\ell}{\iota}$ is the binomial coefficient, we get

$$\begin{aligned} C_{\text{erg}} &= e^{\frac{1}{P}} \int_1^\infty e^{-\frac{t}{P}} \sum_{\ell=0}^{n_r-1} \frac{1}{P^\ell \ell! t} \sum_{\iota=0}^\ell \binom{\ell}{\iota} t^\iota (-1)^{\ell-\iota} dt \\ &= e^{\frac{1}{P}} \sum_{\ell=0}^{n_r-1} \frac{(-1)^\ell}{P^\ell \ell!} \int_1^\infty \frac{e^{-\frac{t}{P}}}{t} dt \\ &\quad + e^{\frac{1}{P}} \sum_{\ell=1}^{n_r-1} \frac{1}{P^\ell \ell!} \sum_{\iota=1}^\ell (-1)^{\ell-\iota} \binom{\ell}{\iota} \int_1^\infty e^{-\frac{t}{P}} t^{\iota-1} dt. \end{aligned} \quad (109)$$

With $u = \frac{t}{P}$, we have

$$\begin{aligned} \int_1^\infty e^{-\frac{t}{P}} t^{\iota-1} dt &= P^\iota \int_{\frac{1}{P}}^\infty e^{-u} u^{\iota-1} du \\ &= (\iota-1)! P^\iota e^{-\frac{1}{P}} \sum_{m=0}^{\iota-1} \frac{1}{m!} \left(\frac{1}{P} \right)^m. \end{aligned} \quad (110)$$

Inserting Eq. (110) into Eq. (109), we obtain

$$\begin{aligned} C_{\text{erg}} &= e^{\frac{1}{P}} E_1 \left(\frac{1}{P} \right) \sum_{\ell=0}^{n_r-1} \frac{(-1)^\ell}{P^\ell \ell!} \\ &\quad + \sum_{\ell=1}^{n_r-1} \frac{1}{P^\ell} \sum_{\iota=1}^\ell \frac{(-1)^{\ell-\iota}}{\iota(\ell-\iota)!} P^\iota \sum_{m=0}^{\iota-1} \frac{1}{m!} \frac{1}{P^m}. \end{aligned} \quad (111)$$

Let $k = \ell - \iota$, the above leads to

$$\begin{aligned} C_{\text{erg}} &= e^{\frac{1}{P}} E_1 \left(\frac{1}{P} \right) \sum_{\ell=0}^{n_r-1} \frac{(-1)^\ell}{P^\ell \ell!} \\ &\quad + \sum_{\ell=1}^{n_r-1} \sum_{k=0}^{\ell-1} \frac{(-1)^k}{(\ell-k)! k!} \sum_{m=0}^{\ell-k-1} \frac{1}{m! P^{k+m}}. \end{aligned} \quad (112)$$

From [6], the ergodic capacity in an $n_t \times 1$ MISO channel with total power constraint P equals the ergodic capacity in a $1 \times n_t$ SIMO channel with total power constraint $\frac{P}{n_t}$. Hence, we obtain Eq. (13) by replacing P with $\frac{P}{n_t}$ and n_r with n_t in Eq. (112).

APPENDIX B

The indefinite integral (antiderivative) of Eq. (45) can be written as

$$\begin{aligned}
\mathcal{R}(s) &= \int e^{-s} \left(\frac{n_t + 1}{s} - 1 \right) \sum_{\ell=0}^{n_t-1} \frac{s^\ell}{\ell!} ds \\
&= (n_t + 1) \int \frac{e^{-s}}{s} ds + (n_t + 1) \int e^{-s} \sum_{\ell=0}^{n_t-1} \frac{s^{\ell-1}}{\ell!} ds \\
&\quad - \int e^{-s} \sum_{\ell=0}^{n_t-1} \frac{s^\ell}{\ell!} ds \\
&= (n_t + 1) \int \frac{e^{-s}}{s} ds + \sum_{\ell=0}^{n_t-1} \frac{1}{\ell!} \left((n_t + 1) \int s^{\ell-1} e^{-s} ds \right. \\
&\quad \left. - \int s^\ell e^{-s} ds \right). \tag{113}
\end{aligned}$$

The definite integral of $\mathcal{R}(s)$ over the interval $[s_0, \infty]$ is given by

$$\begin{aligned}
[\mathcal{R}(s)]_{s_0}^{\infty} &= (n_t + 1) \int_{s_0}^{\infty} \frac{e^{-s}}{s} ds + \sum_{\ell=0}^{n_t-1} \frac{1}{\ell!} \left(\right. \\
&\quad \left. (n_t + 1) \int_{s_0}^{\infty} s^{\ell-1} e^{-s} ds - \int_{s_0}^{\infty} s^\ell e^{-s} ds \right) \\
&= (n_t + 1) \mathbf{E}_1(s_0) + \sum_{\ell=0}^{n_t-1} \frac{1}{\ell!} \left(\right. \\
&\quad \left. (n_t + 1) (\ell - 1)! e^{-s_0} \sum_{k=0}^{\ell-1} \frac{s_0^k}{k!} - \ell! e^{-s_0} \sum_{k=0}^{\ell} \frac{s_0^k}{k!} \right) \\
&= (n_t + 1) \mathbf{E}_1(s_0) - e^{-s_0} + e^{-s_0} \sum_{\ell=1}^{n_t-1} \frac{1}{\ell!} \left(\right. \\
&\quad \left. - s_0^\ell + (n_t + 1 - \ell) (\ell - 1)! \sum_{k=0}^{\ell-1} \frac{s_0^k}{k!} \right). \tag{114}
\end{aligned}$$

The definite integral of $\mathcal{R}(s)$ over the interval $[s_0, s_1]$ can be written as $[\mathcal{R}(s)]_{s_0}^{\infty} - [\mathcal{R}(s)]_{s_1}^{\infty}$. Therefore, defining

$$\begin{aligned}
\mathcal{R}(s) &\triangleq - (n_t + 1) \mathbf{E}_1(s) + e^{-s} \\
&\quad + e^{-s} \sum_{\ell=1}^{n_t-1} \frac{1}{\ell!} \left(s^\ell - (n_t + 1 - \ell) (\ell - 1)! \sum_{k=0}^{\ell-1} \frac{s^k}{k!} \right), \tag{115}
\end{aligned}$$

and inserting into Eq. (114) leads to the conclusion that

$$[\mathcal{R}(s)]_{s_0}^{s_1} = \mathcal{R}(s_1) - \mathcal{R}(s_0). \tag{116}$$

REFERENCES

- [1] E. Biglieri, J. Proakis, and S. Shamai, "Fading channels: Information theoretic and communication aspects," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2619–2692, 1998.
- [2] G. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs. Tech. J.*, vol. 1, no. 2, pp. 41–59, 1996.
- [3] V. Tarokh, N. Seshadri, and A. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, no. 2, pp. 744–765, 1998.
- [4] V. Tarokh, H. Jafarkhani, and A. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1456–1467, 1999.
- [5] P. Wolniansky, G. Foschini, G. Golden, and R. Valenzuela, "V-BLAST: An architecture for realizing very high data rates over the rich-scattering wireless channel," in *URSI IEEE Int. Symp. Signals, Systems, and Electronics, ISSSE*, 1998, pp. 295–300.
- [6] E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecommun.*, vol. 10, no. 6, pp. 585–595, 1999.
- [7] T. Marzetta and B. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 45, no. 1, pp. 139–157, 1999.
- [8] G. Foschini and M. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless personal commun.*, vol. 6, no. 3, pp. 311–335, 1998.
- [9] L. H. Ozarow, S. Shamai, and A. D. Wyner, "Information theoretic considerations for cellular mobile radio," *IEEE Trans. Veh. Technol.*, vol. 43, pp. 359–378, 1994.
- [10] E. Jorswieck and H. Boche, "Outage probability in multiple antenna systems," *Europ. Trans. Telecommun.*, vol. 18, no. 3, pp. 217–233, 2007.
- [11] N. Ahmed and R. Baraniuk, "Throughput measures for delay-constrained communications in fading channels," in *Proc. Allertin Conf. Commun., Control, and Computing*, vol. 41, no. 3. Citeseer, 2003, pp. 1496–1505.
- [12] S. R. Mirghaderi, A. Bayesteh, and A. Khandani, "On the maximum achievable rates in wireless multicast networks," in *Proc. IEEE Int. Symp. Inform. Theory, ISIT*, 2007, pp. 1201–1205.
- [13] S. Shamai and A. Steiner, "A broadcast approach for a single-user slowly fading MIMO channel," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2617–2634, 2003.
- [14] A. Steiner and S. Shamai, "Single-user broadcasting protocols over a two-hop relay fading channel," *IEEE Trans. Inform. Theory*, vol. 52, no. 11, pp. 4821–4838, 2006.
- [15] V. Pourahmadi, A. Bayesteh, and A. Khandani, "Multilevel coding strategy for two-hop single-user networks," in *24th Biennial Symp. Commun.*, 2008, pp. 115–119.
- [16] A. Steiner and S. Shamai, "Broadcast cooperation strategies for two collocated users," *IEEE Trans. Inform. Theory*, vol. 53, no. 10, pp. 3394–3412, 2007.
- [17] M. Zamani and A. Khandani, "On the maximum achievable rates in the decode-forward diamond channel," in *Proc. IEEE Int. Symp. Inform. Theory, ISIT*, 2011, pp. 1594–1598.
- [18] P. Minero and D. Tse, "A broadcast approach to multiple access with random states," in *Proc. IEEE Int. Symp. Inform. Theory, ISIT*, 2007, pp. 2566–2570.
- [19] Z. Wang and G. Giannakis, "Outage mutual information of space-time MIMO channels," *IEEE Trans. Inform. Theory*, vol. 50, no. 4, pp. 657–662, 2004.
- [20] B. Hochwald, T. Marzetta, and V. Tarokh, "Multiple-antenna channel hardening and its implications for rate feedback and scheduling," *IEEE Trans. Inform. Theory*, vol. 50, no. 9, pp. 1893–1909, 2004.
- [21] R. Corless, G. Gonnet, D. Hare, D. Jeffrey, and D. Knuth, "On the Lambert W function," *Advances in Computational mathematics*, vol. 5, no. 1, pp. 329–359, 1996.
- [22] I. Gradshteyn, I. Ryzhik, and A. Jeffrey, *Table of integrals, series, and products*. Academic Pr, 2000.

- [23] A. Tulino and S. Verdú, *Random matrix theory and wireless communications*. Now Publishers Inc, 2004.
- [24] A. Papoulis and S. Pillai, *Probability, random variables, and stochastic processes*. McGraw-hill New York, 2002.
- [25] H. Boche and E. Jorswieck, "On the ergodic capacity as a function of the correlation properties in systems with multiple transmit antennas without CSI at the transmitter," *IEEE Trans. Commun.*, vol. 52, no. 10, pp. 1654–1657, 2004.
- [26] S. Shamai, "A broadcast strategy for the gaussian slowly fading channel," in *Proc. IEEE Int. Symp. Inform. Theory, ISIT*, 1997, p. 150.
- [27] T. Cover and J. Thomas, *Elements of information theory*. John Wiley & Sons, 2006.
- [28] A. Steiner and S. Shamai, "Multi-layer broadcasting over a block fading MIMO channel," *IEEE Trans. Wireless Commun.*, vol. 6, no. 11, pp. 3937–3945, 2007.
- [29] E. Visotsky and U. Madhow, "Space-time transmit precoding with imperfect feedback," *IEEE Trans. Inform. Theory*, vol. 47, no. 6, pp. 2632–2639, 2002.
- [30] I. Gelfand and S. Fomin, "Calculus of variations. Revised English edition translated and edited by Richard A. Silverman," 1963.
- [31] J. Silverstein and Z. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," *J. Multivariate analysis*, vol. 54, no. 2, pp. 175–192, 1995.
- [32] C. Chuah, D. Tse, J. Kahn, and R. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Trans. Inform. Theory*, vol. 48, no. 3, pp. 637–650, 2002.
- [33] T. Anderson, *An introduction to multivariate statistical analysis, 3rd edn.* John Wiley & Sons, 2003.
- [34] R. Muirhead, *Aspects of multivariate statistical theory*. John Wiley & Sons, 1982.
- [35] T. Ratnarajah and R. Vaillancourt, "Complex singular wishart matrices and applications," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 399–411, 2005.
- [36] A. Kshirsagar, "Bartlett decomposition and wishart distribution," *The Annals of Mathematical Statistics*, vol. 30, no. 1, pp. 239–241, 1959.
- [37] A. Goldsmith, S. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," *IEEE J. Select. Areas Commun.*, vol. 21, no. 5, pp. 684–702, 2003.
- [38] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. select. areas commun.*, vol. 16, no. 8, pp. 1451–1458, 1998.